Data-based approach for time-correlated closures of turbulence models

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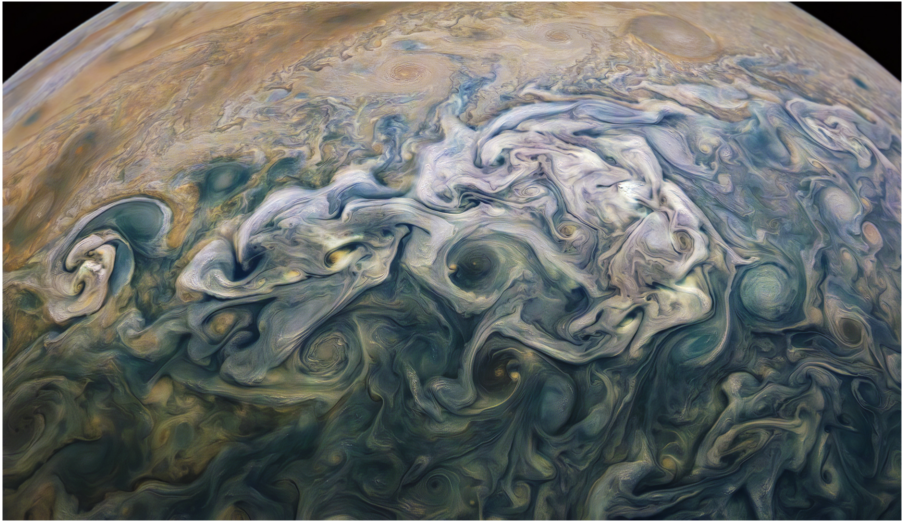
Turbulence



Typical values

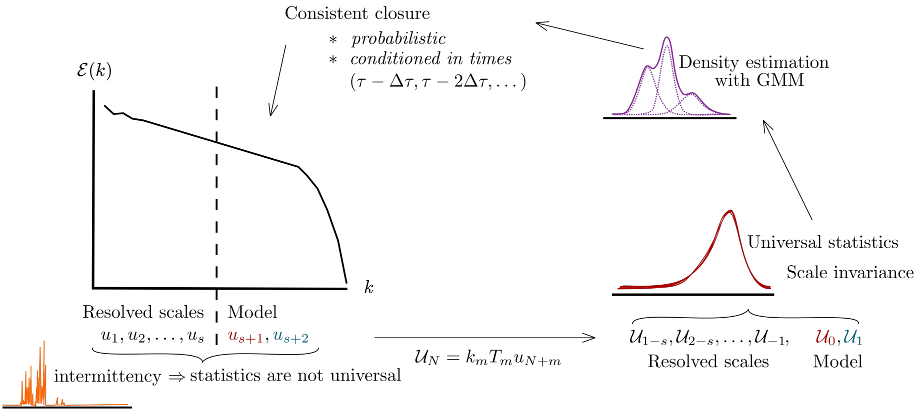
$$Re = \frac{UL}{\nu}, \quad \eta \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4}, \quad \epsilon \sim \frac{U^3}{L}$$

For a grid with spacing smaller than the Kolmogorov length scale:

$$N = \left(\frac{L}{\eta}\right)^3 \sim Re^{9/4} \tag{1}$$

- Cup of coffee: $Re = 10^4$, grid must have 10^9 points (8GB per array)
- Moving car: $Re = 10^9$, grid must have 10^{16} points (80000000GB per array)

Road map



Consistent closure
* probabilistic
* conditioned in times
$(\tau - \Delta\tau, \tau - 2\Delta\tau, \dots)$
Density estimation
with GMM
$\mathcal{E}(k)$
k
Resolved scales
Model
$u_1, u_2, \dots, u_s$
u_{s+1}, u_{s+2}
intermittency $\Rightarrow$ statistics are not universal
$\mathcal{U}_N = k_m T_m u_{N+m}$
Universal statistics
Scale invariance
$\mathcal{U}_{1-s}, \mathcal{U}_{2-s}, \dots, \mathcal{U}_{-1},$
$\mathcal{U}_0, \mathcal{U}_1$
Resolved scales
Model

Models of turbulence

N.S.

$$\partial_t u_i(\mathbf{n}) = -i\frac{2\pi n_j}{L}\sum_{\mathbf{n'}}\left(\delta_{il} - \frac{n_i n_l'}{n_j n_j}\right)u_j(\mathbf{n}')u_l(\mathbf{n}-\mathbf{n}') - \nu k_j k_j u_i(\mathbf{n}) + f_i(\mathbf{n}). \tag{2}$$

L'vov et al. (1995) and Gledzer, Ohkitani, Yamada (1973,1988)

$$\frac{du_n}{dt} = i\left(ak_{n+1}u_{n+2}u_{n+1}^* + bk_n u_{n+1}u_{n-1}^* + ck_{n-1}u_{n-1}u_{n-2}\right) - \nu k_n^2 u_n + f_n \tag{3}$$

About Sabra

Parameters

- $a = 1$, $b = -\frac{1}{2}$, $c = \frac{1}{2}$, $\lambda = 2$, $k_0 = 1$ and $k_n = k_0 \lambda^n$:
- Inviscid invariants:

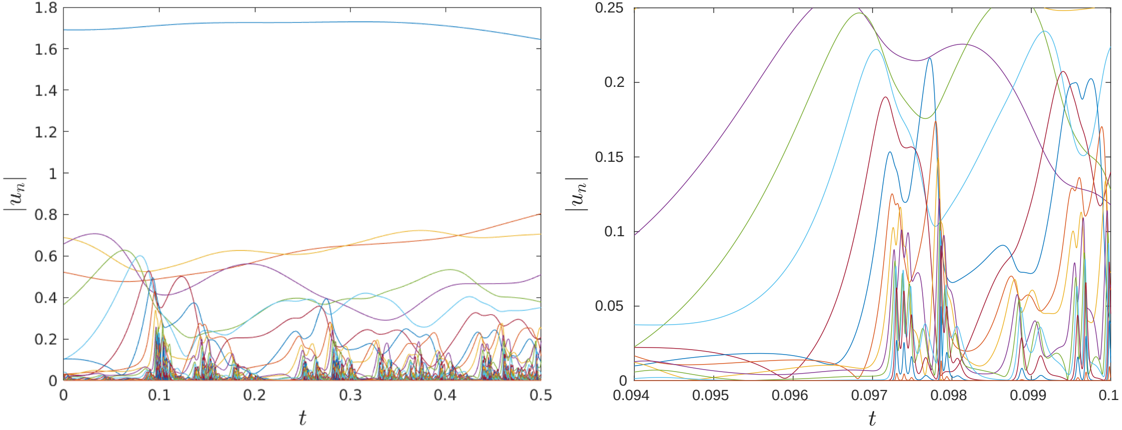
$$E = \sum |u_n|^2 \tag{4}$$

$$H = \sum (-1)^n k_n |u_n|^2 \tag{5}$$

Numerics

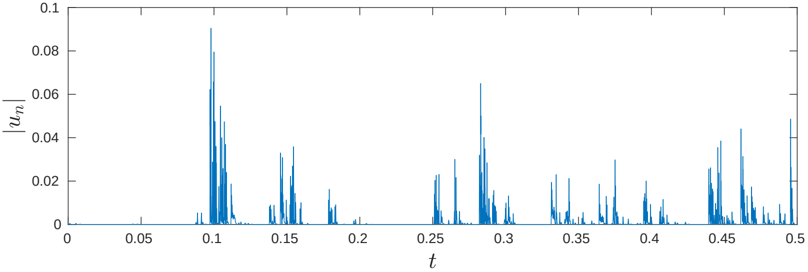
- $n = 30$
- $u_{-1} = u_0 = u_{n+1} = u_{n+2} = 0$ (for now)
- $\nu = 10^{-8}$, $f_1 = 1 + i$, $u_n(0) = k^{-1/3} e^{iq_n}$ for $n = 1, 2$

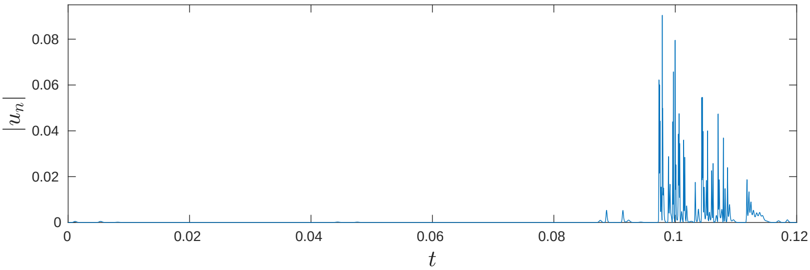
Numerics



1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
$|u_n|$
0
0.1
0.2
0.3
0.4
0.5
t
0.25
0.2
0.15
0.1
0.05
0
$|u_n|$
0.094
0.095
0.096
0.097
0.098
0.099
0.1
t

Intermittency





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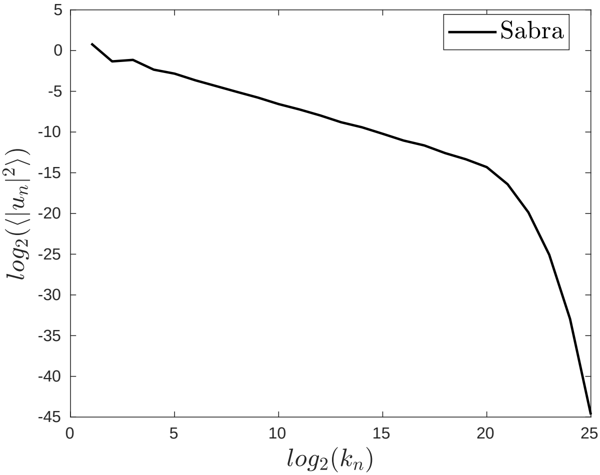
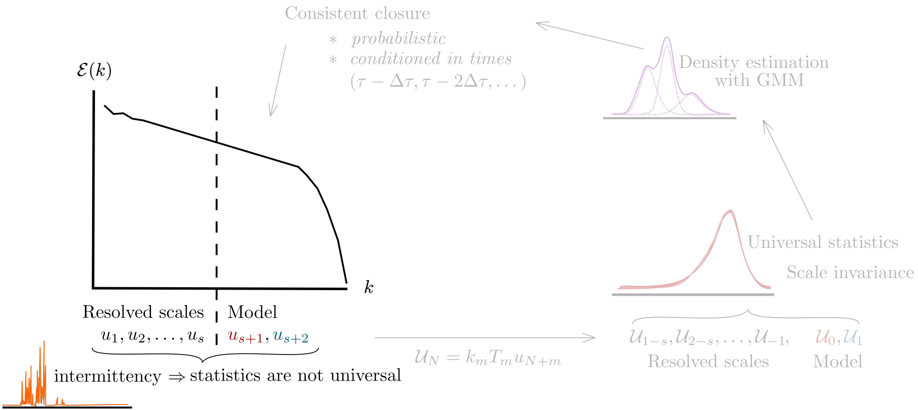


Sabra
5
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
0
5
10
15
20
25
$log_2(\langle|u_n|^2\rangle)$
$log_2(k_n)$

Road map



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Multipliers

$$z_n = w_n e^{i\Delta_n}, \tag{6}$$

$$w_n = \left| \frac{u_n}{u_{n-1}} \right|, \tag{7}$$

$$\Delta_n = \arg(u_n) - \arg(u_{n-1}) - \arg(u_{n-2}), \tag{8}$$

The need for time conditioning

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Optimal subgrid scheme for shell models of turbulence

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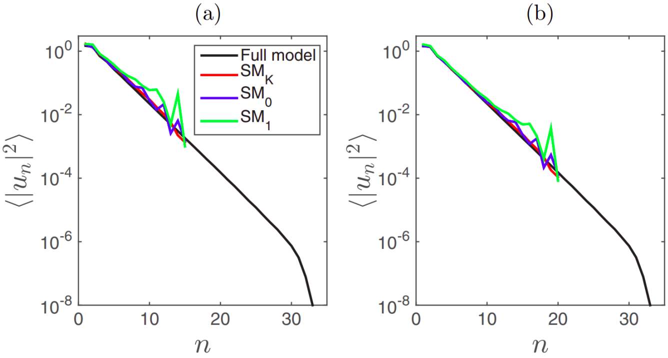


[Kolmogorov, 1962]
[Benzi, 1993]

Rescaled Model

Choose a reference shell m [Mailybaev, 2021]

$$T_m(t) = \left(k_0^2 U^2 + \sum_{n<m} k_n^2 |u_n|^2 \right)^{-1/2} \tag{9}$$

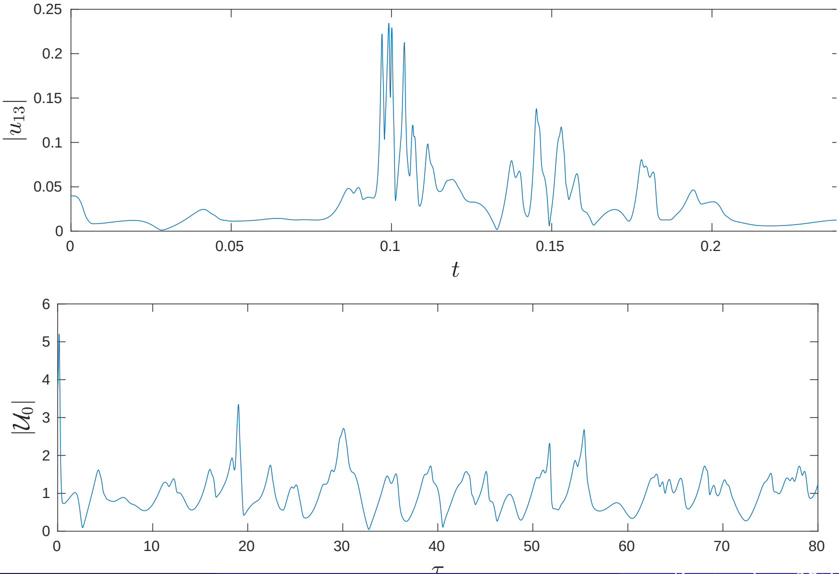
Define the change of variables

$$\tau = \int_0^t \frac{dt'}{T_m(t')} \tag{10}$$

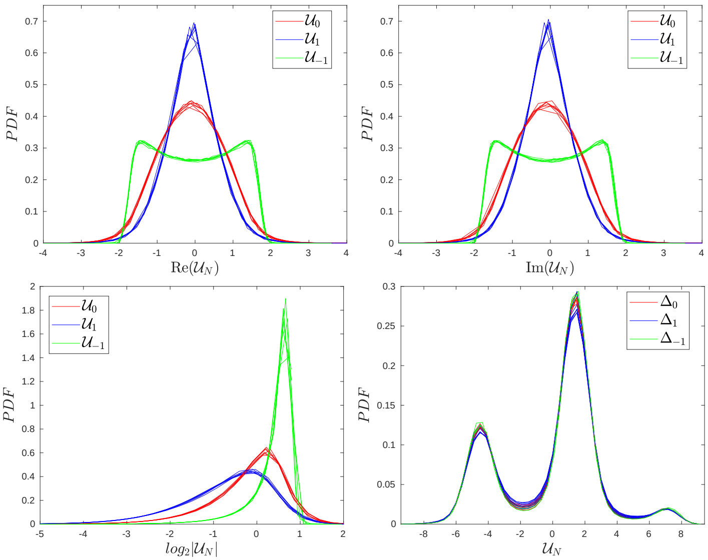
$$\mathcal{U}_N = k_m T_m(t) u_{N+m}(t) \tag{11}$$

- Shrinks long periods of time where nothing happens
- Stretches short periods of time where a lot happens
- Gives a new sense of symmetry

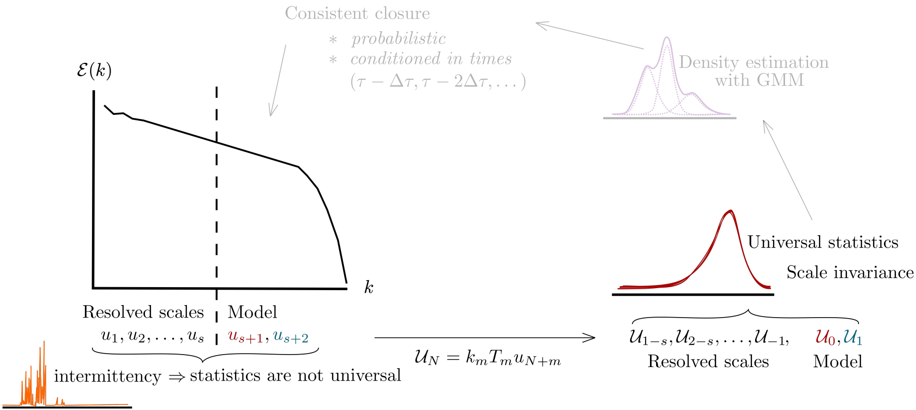
Rescaled Model



Hidden Symmetry



Road map



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Rescaled Complete System

$$\frac{d\mathcal{U}_N}{d\tau} = i(k_{N+1}\mathcal{U}_{N+2}\mathcal{U}^*_{N+1} - \frac{1}{2}k_N\mathcal{U}_{N+1}\mathcal{U}^*_{N-1} + \frac{1}{2}k_{N-1}\mathcal{U}_{N-1}\mathcal{U}_{N-2}) \\ + (\xi + \xi_f)\mathcal{U}_N + \nu k_m^2 T_m\Big(-k_N^2 + \sum_{J<0} k_J^4 |\mathcal{U}_J|^2\Big)\mathcal{U}_N + T_m^2 k_m f_{N+m} \tag{12}$$

where

$$\xi = \sum_{N<0} k_N^3 \,\mathrm{Im}\left(2\mathcal{U}^*_N\mathcal{U}^*_{N+1}\mathcal{U}_{N+2} - \frac{1}{2} \qquad \mathcal{U}^*_{N-1}\mathcal{U}^*_N\mathcal{U}_{N+1} - \frac{1}{4}\mathcal{U}^*_{N-1}\mathcal{U}_N\mathcal{U}^*_{N-2}\right) \tag{13}$$

$$\xi_f = -T_m^2 \sum_{N<0} k_{N+m}k_N \,\mathrm{Re}\left(\mathcal{U}^*_N f_{N+m}\right) \tag{14}$$

$$T_m = \frac{1}{k_0 U}\left(1 - \sum_{N<0} k_N^2 |\mathcal{U}_N|^2\right)^{1/2} \tag{15}$$

Closure for Reduced Models

- Choose a shell s in the inertial range

$$\frac{du_n}{dt} = i\big(ak_{n+1}u_{n+2}{u_{n+1}}^* + bk_n u_{n+1}u_{n-1}^* + ck_{n-1}u_{n-1}u_{n-2}\big) - \nu k_n^2 u_n + f_n \tag{16}$$

for $n = 1, \ldots, s$

- Choose $m = s + 1$

$$\frac{d\mathcal{U}_N}{d\tau} = i(k_{N+1}\mathcal{U}_{N+2}{\mathcal{U}_{N+1}}^* - \frac{1}{2}k_N\mathcal{U}_{N+1}\mathcal{U}_{N-1}^* + \frac{1}{2}k_{N-1}\mathcal{U}_{N-1}\mathcal{U}_{N-2}) \\ + (\xi + \xi_f)\mathcal{U}_N + \nu k_m^2 T_m\big(-k_N^2 + \sum_{J<0} k_J^4|\mathcal{U}_J|^2\big)\mathcal{U}_N + T_m^2 k_m f_{N+m} \tag{17}$$

For $N = 1 - s, \cdots, -1$.

Main Message

We need expressions for u_{s+1} and u_{s+2}.

\- or -

We need expressions for $\mathcal{U}_0$ and $\mathcal{U}_1$.

Kolmogorov's Closure

[Biferale, Mailybaev, Parisi, 2017]

$$\frac{|u_{s+2}|}{|u_{s+1}|} = \frac{|u_{s+1}|}{|u_s|} = \lambda^{-1/3}, \tag{18}$$

$$\Delta_{s+1} = \Delta_{s+2} = \frac{\pi}{2}. \tag{19}$$

Choosing the the shell velocities' phases θ_n as to satisfy $\theta_n = \theta_{n-1} + \theta_{n-2}$,

$$u_{s+1} = |u_s|\lambda^{-1/3} e^{i(\frac{\pi}{2}+\theta_s+\theta_{s-1})}, \tag{20}$$

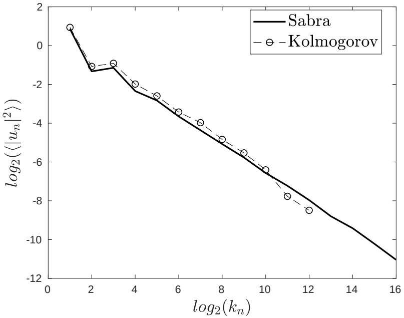
$$u_{s+2} = |u_{s+1}|\lambda^{-1/3} e^{i(\frac{\pi}{2}+\theta_{s+1}+\theta_s)}. \tag{21}$$

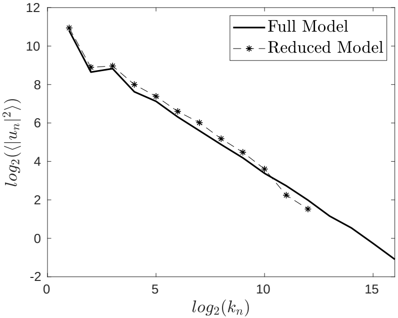
Kolmogorov's Closure

$$\mathcal{U}_0 = |\mathcal{U}_{-1}|\lambda^{-1/3} e^{i(\frac{\pi}{2}+\alpha_{-1}+\alpha_{-2})}, \tag{22}$$

$$\mathcal{U}_1 = |\mathcal{U}_0|\lambda^{-1/3} e^{i(\frac{\pi}{2}+\alpha_0+\alpha_{-1})} \tag{23}$$

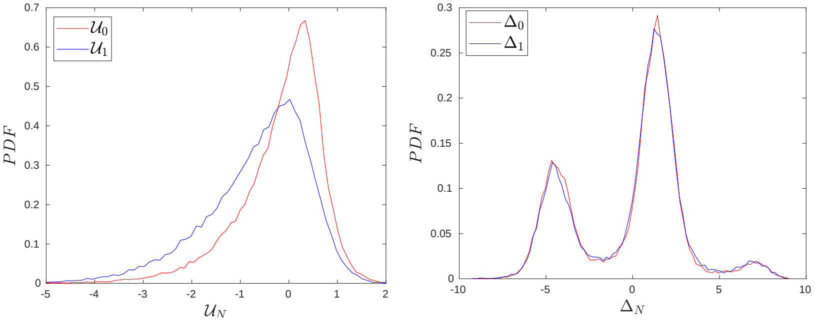
$$\alpha_N = \arg(\mathcal{U}_N) = \theta_{N+m}$$



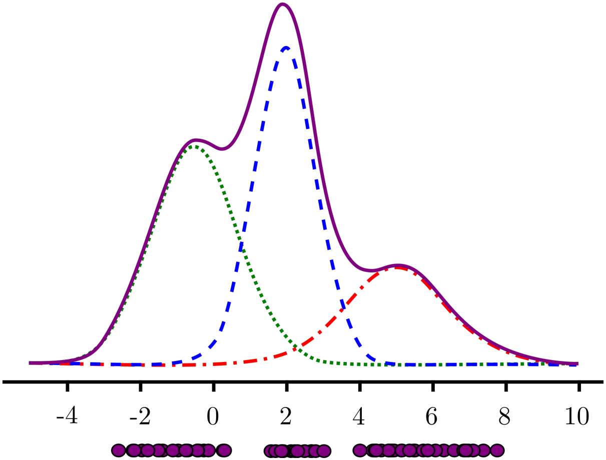


Probabilistic approach

How do we go about making this *probabilistic*?



Gaussian Mixture Models



Gaussian Mixture Models

Say we have N data samples from a distribution we don't know.

$$p(\mathbf{x}) = \sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \tag{24}$$

$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right) \tag{25}$$

$$\sum_{n=1}^{k} \pi_k = 1. \tag{26}$$

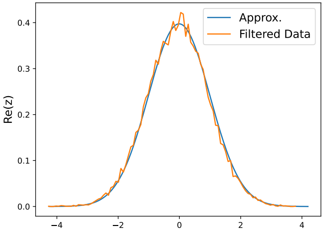
Gaussian Mixture Models

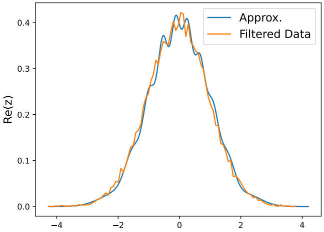
$$\gamma(z_{nk}) = \frac{\pi_k \mathcal{N}(\mathbf{x}_n | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}{\sum_{j=1}^{k} \pi_j \mathcal{N}(\mathbf{x}_n | \boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)} \tag{27}$$

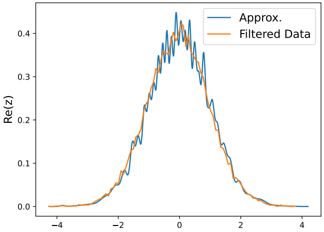
$$\mathcal{Q}(\boldsymbol{\theta}^*, \boldsymbol{\theta}) = \sum_{n=1}^{N} \sum_{k=1}^{K} \gamma(z_{nk}) (\ln \pi_k^* + \ln \mathcal{N}(\mathbf{x}_n | \boldsymbol{\mu}_k^*, \boldsymbol{\Sigma}_k^*)) + \lambda \left(\sum_{k=1}^{K} \pi_k^* - 1 \right) \tag{28}$$

$$\widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}^*} \mathcal{Q}(\boldsymbol{\theta}^*, \boldsymbol{\theta}) \tag{29}$$

Convergence







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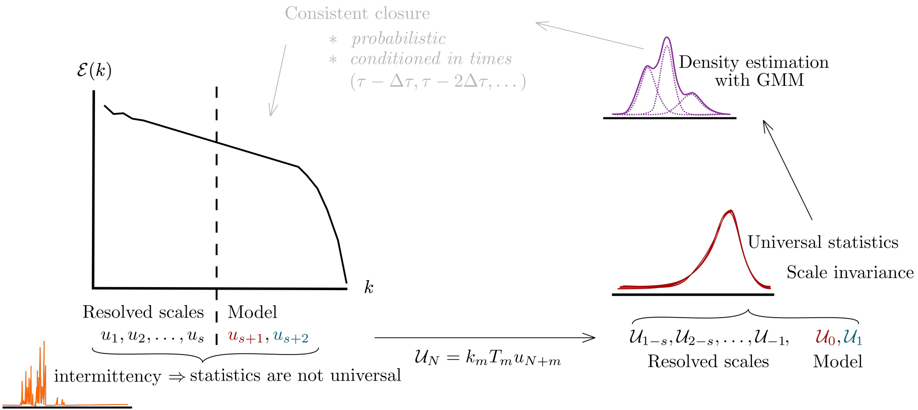

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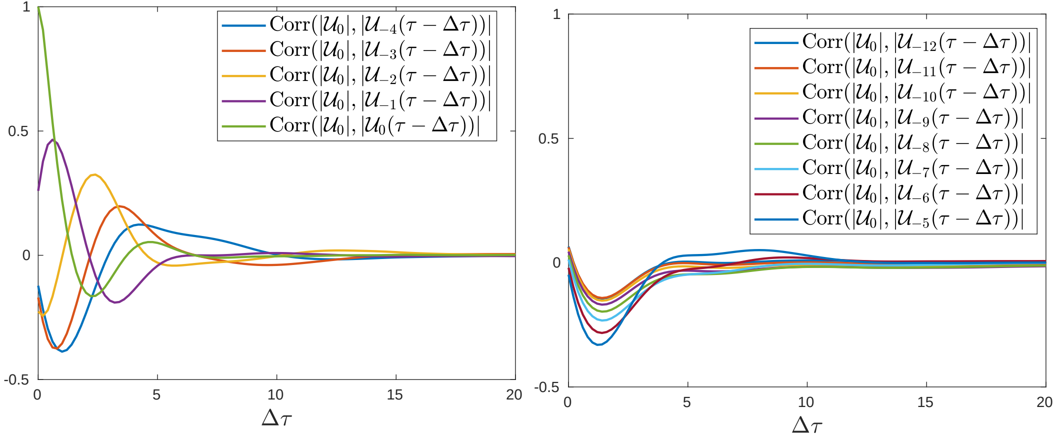
Time Correlation



Single-time closures

Only modules:

$$\mathcal{U}_0 = 2^{z_0} e^{i(\frac{\pi}{2}+\alpha_{-1}+\alpha_{-2})}, \tag{30}$$

$$\mathcal{U}_1 = 2^{z_1} e^{i(\frac{\pi}{2}+\alpha_{0}+\alpha_{-1})}, \tag{31}$$

$$\mathbf{z} = (z_0, z_1) \sim g(\mathbf{z}). \tag{32}$$

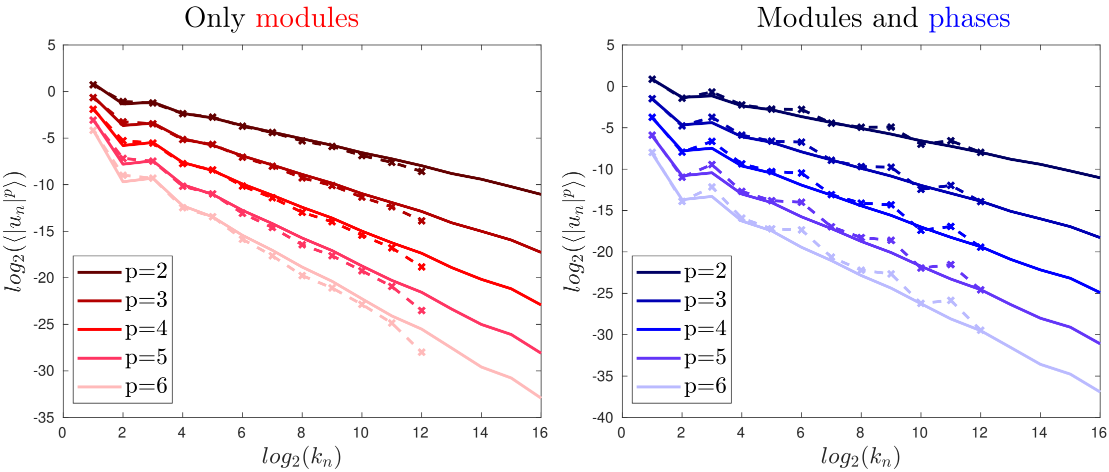
Modules and phases:

$$\mathcal{U}_0 = 2^{z_0} e^{i(z_1+\alpha_{-1}+\alpha_{-2})}, \tag{33}$$

$$\mathcal{U}_1 = 2^{z_2} e^{i(z_3+\alpha_{0}+\alpha_{-1})}, \tag{34}$$

$$\mathbf{z} = (z_0, z_1, z_2, z_3) \sim g(\mathbf{z}). \tag{35}$$

Single-time closures



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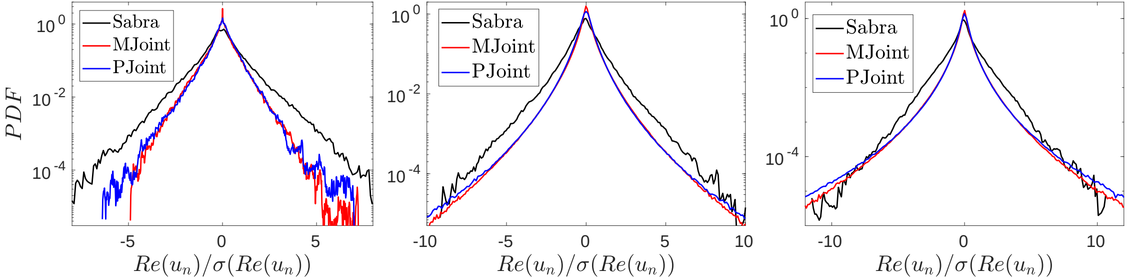
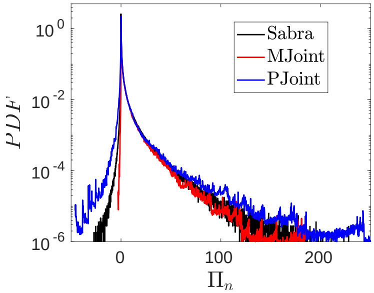
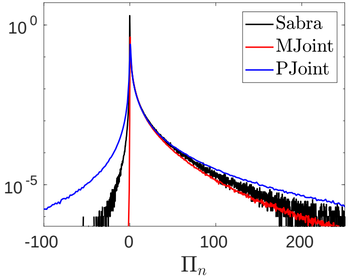


n=12 n=13 n=14

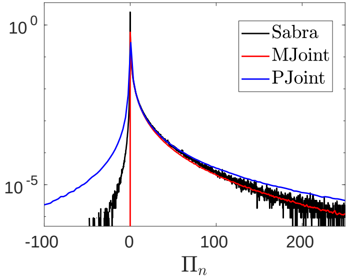
Single-time closures



n=9



n=11



n=12

Time conditioning

Conditioning to modules of three closest shells:

$$\mathcal{U}_0 = 2^{z_0} e^{i(\frac{\pi}{2}+\alpha_{-1}+\alpha_{-2})}, \tag{36}$$

$$\mathcal{U}_1 = 2^{z_1} e^{i(\frac{\pi}{2}+\alpha_{0}+\alpha_{-1})}, \tag{37}$$

$$\mathbf{z} = (z_0, z_1) \sim g(\mathbf{z}| \log_2 |\mathcal{U}_{-3}(\tau - \Delta\tau)|, \log_2 |\mathcal{U}_{-2}(\tau - \Delta\tau)|, \log_2 |\mathcal{U}_{-1}(\tau - \Delta\tau)|), \tag{38}$$

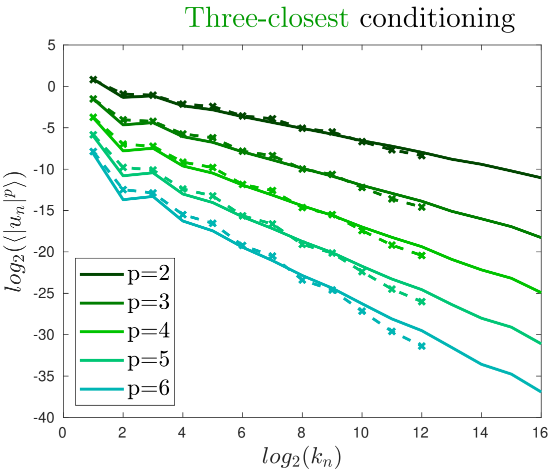
Conditioning to modules and phases of themselves:

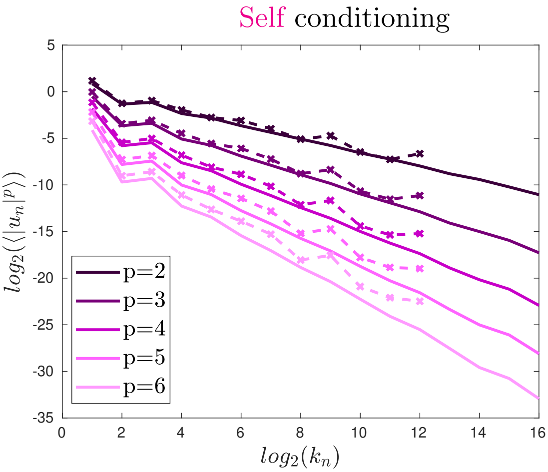
$$\mathcal{U}_0 = 2^{z_0} e^{i(z_1+\alpha_{-1}+\alpha_{-2})}, \tag{39}$$

$$\mathcal{U}_1 = 2^{z_2} e^{i(z_3+\alpha_{0}+\alpha_{-1})}, \tag{40}$$

$$\mathbf{z} = (z_0, z_1, z_2, z_3) \sim g(\mathbf{z}| \log_2 |\mathcal{U}_0(\tau - \Delta\tau)|, \Delta_0, \log_2 |\mathcal{U}_1(\tau - \Delta\tau)|, \Delta_1). \tag{41}$$

Time conditioning, $\Delta\tau = 2.4$





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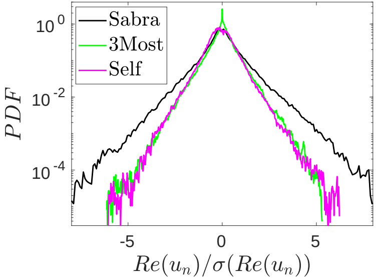
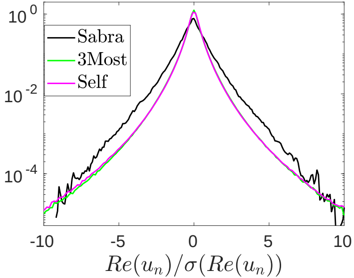
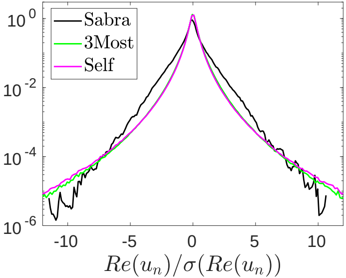


n=12

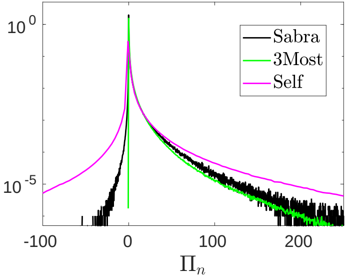


n=13



n=14

Time conditioning



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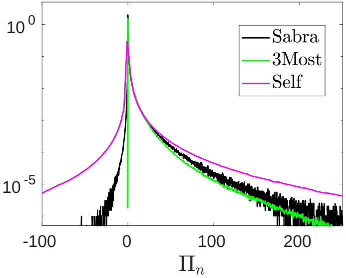
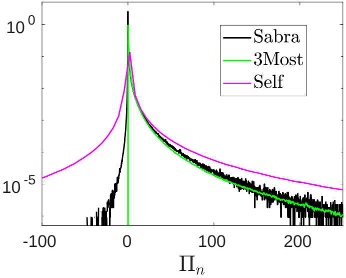


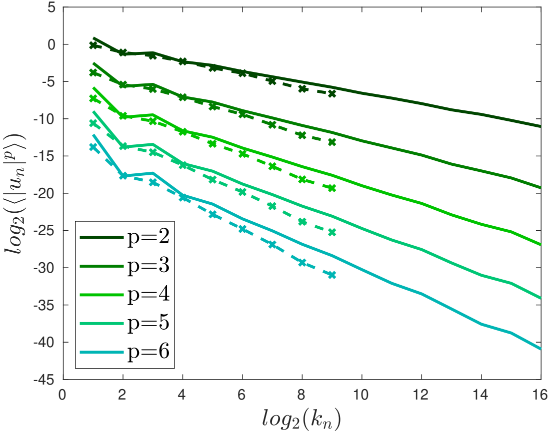
n=11



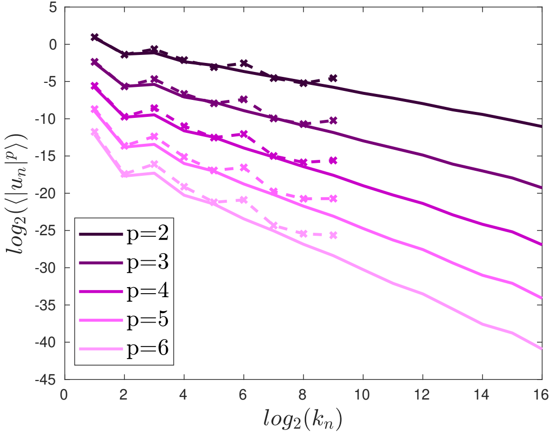
n=12

Time conditioning, $\Delta\tau = 2.4$

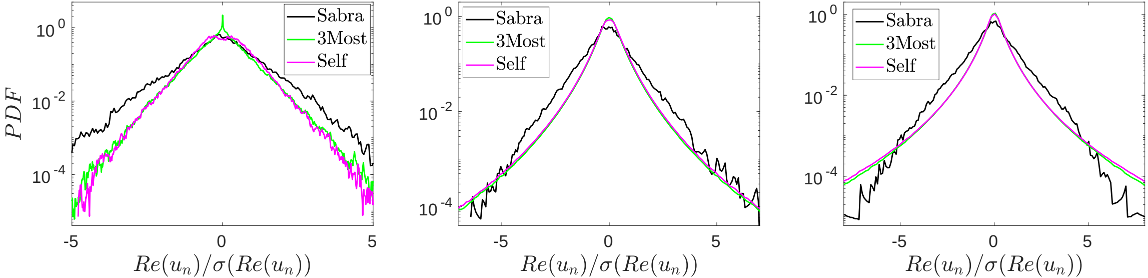
Three-closest conditioning



Self conditioning

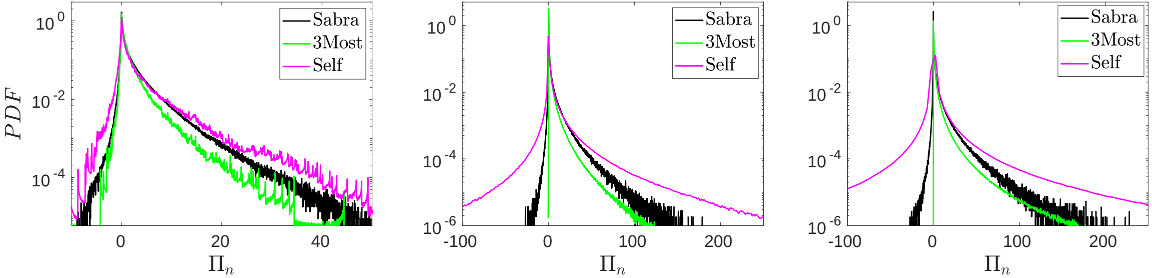


Time conditioning



n=9 n=10 n=11

Time conditioning

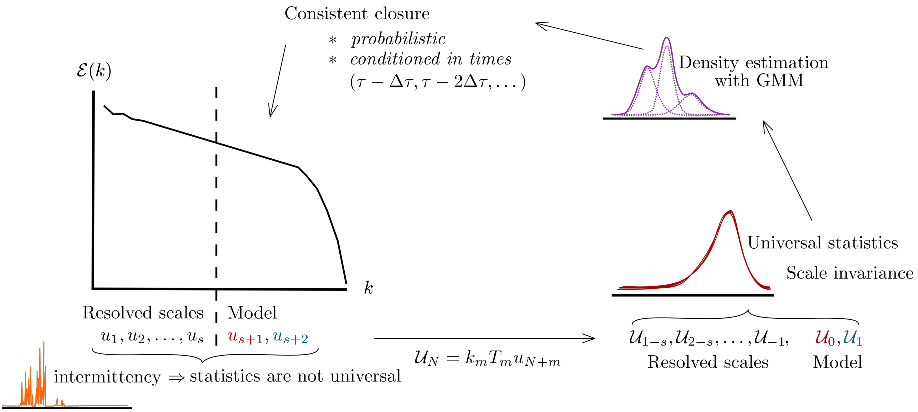


n=4 n=8 n=9

All results

	$\lvert\mathcal{U}_0\rvert$	$\lvert\mathcal{U}_1\rvert$	Δ_0	Δ_1	Conditioning
Half closure	2^{z_0}	$\lvert\mathcal{U}_0\rvert\lambda^{-1/3}$	$\pi/2$	$\pi/2$	$\times$
Joint	2^{z_0}	2^{z_1}	$\pi/2$	$\pi/2$	$\times$
Simple cond	2^{z_0}	$\lvert\mathcal{U}_0\rvert\lambda^{-1/3}$	$\pi/2$	$\pi/2$	$\lvert\mathcal{U}_{-1}\rvert$ at $\tau-\Delta\tau$
Joint cond	2^{z_0}	2^{z_1}	$\pi/2$	$\pi/2$	$\lvert\mathcal{U}_{-1}\rvert$ at $\tau-\Delta\tau$
3-Clos	2^{z_0}	2^{z_1}	$\pi/2$	$\pi/2$	$\lvert\mathcal{U}_{-3}\rvert, \lvert\mathcal{U}_{-2}\rvert, \lvert\mathcal{U}_{-1}\rvert$ at $\tau-\Delta\tau$
3-Clos 9	2^{z_0}	2^{z_1}	$\pi/2$	$\pi/2$	$\lvert\mathcal{U}_{-3}\rvert, \lvert\mathcal{U}_{-2}\rvert, \lvert\mathcal{U}_{-1}\rvert$ at $\tau-\Delta\tau$
Long	2^{z_0}	2^{z_1}	$\pi/2$	$\pi/2$	$\lvert\mathcal{U}_{-s}\rvert, \ldots, \lvert\mathcal{U}_{-1}\rvert$ at $\tau-\Delta\tau$
Joint phases	2^{z_0}	2^{z_2}	z_1	z_3	$\times$
Joint cond phases	2^{z_0}	2^{z_2}	z_1	z_3	$\lvert\mathcal{U}_{-1}\rvert, \Delta{-1}$ at $\tau-\Delta\tau$
Self	2^{z_0}	2^{z_2}	z_1	z_3	$\lvert\mathcal{U}_0\rvert, \lvert\mathcal{U}_1\rvert, \Delta_0, \Delta_1$ at $\tau-\Delta\tau$
Self 9	2^{z_0}	2^{z_2}	z_1	z_3	$\lvert\mathcal{U}_0\rvert, \lvert\mathcal{U}_1\rvert, \Delta_0, \Delta_1$ at $\tau-\Delta\tau$
Global	2^{z_0}	2^{z_2}	z_1	z_3	$\lvert\mathcal{U}_{-2}\rvert, \lvert\mathcal{U}_{-1}\rvert, \Delta_{-2}, \Delta_{-1}$ at τ $\lvert\mathcal{U}_{-2}\rvert, \lvert\mathcal{U}_{-1}\rvert, \lvert\mathcal{U}_0\rvert, \lvert\mathcal{U}_1\rvert, \Delta_{-2}, \Delta_{-1}, \Delta_0, \Delta_1$ at $\tau-\Delta\tau$
2 Times	2^{z_0}	2^{z_2}	z_1	z_3	$\lvert\mathcal{U}_{-2}\rvert, \lvert\mathcal{U}_{-1}\rvert, \Delta_{-2}, \Delta_{-1}$ at τ $\lvert\mathcal{U}_{-2}\rvert, \lvert\mathcal{U}_{-1}\rvert, \lvert\mathcal{U}_0\rvert, \lvert\mathcal{U}_1\rvert, \Delta_{-2}, \Delta_{-1}, \Delta_0, \Delta_1$ at $\tau-\Delta\tau$ $\lvert\mathcal{U}_{-2}\rvert, \lvert\mathcal{U}_{-1}\rvert, \lvert\mathcal{U}_0\rvert, \lvert\mathcal{U}_1\rvert, \Delta_{-2}, \Delta_{-1}, \Delta_0, \Delta_1$ at $\tau-2\Delta\tau$

Road map


Consistent closure
* probabilistic
* conditioned in times
$(\tau - \Delta\tau, \tau - 2\Delta\tau, \dots)$
Density estimation with GMM
$\mathcal{E}(k)$
k
Universal statistics
Scale invariance
Resolved scales
Model
$u_1, u_2, \dots, u_s$
u_{s+1}, u_{s+2}
intermittency $\Rightarrow$ statistics are not universal
$\mathcal{U}_N = k_m T_m u_{N+m}$
$\mathcal{U}_{1-s}, \mathcal{U}_{2-s}, \dots, \mathcal{U}_{-1},$
Resolved scales
$\mathcal{U}_0, \mathcal{U}_1$
Model

Conclusion

- We have systematically written data-based closures (**probabilistic** and **time-correlated**) for shell models
- Time-correlated closures are working with the present approach
- They may work much better with a better approximation of the densities (other ML tools)
- High-dimensional problems are a significant step in this ladder
- This framework reduced black-box aspects

Thank you!